Partial differentiation

Frames 1 to 81

Learning outcomes

When you have completed this Programme you will be able to:

- Derive the expression for a small increment in an expression of two real variables using Taylor's theorem
- Apply the notion of small increments in expressions in two and three real variables to a variety of problems
- Determine the rate of change with respect to time of an expression involving two or three real variables
- Differentiate implicit functions
- Determine first and second derivatives involving change of variables in expressions of two real variables
- Use the Jacobian to obtain the derivatives of inverse functions of two real variables
- Locate and identify maxima, minima and saddle points of functions of two real variables
- Solve problems where the independent variables are constrained by using the method of Lagrange undetermined multipliers for functions of two and three real variables

Prerequisite: Engineering Mathematics (Fifth Edition)
Programmes 9 Differentiation applications 2, 10 Partial differentiation 1 and 11 Partial differentiation 2

Small increments

Taylor's theorem for one independent variable

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Taylor's theorem expands $f(x+h)$ in terms of $f(x)$, powers of h and successive derivatives of $f(x)$, and can be stated as

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots + \frac{h^n}{n!}f^{(n)}(x) + \ldots$$

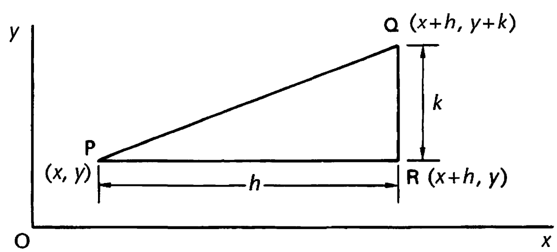
where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$. You will also, no doubt, remember that, by putting $x = 0$ in the result and then letting $h = x$, we obtain Maclaurin's series

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + \ldots + \frac{h^n}{n!}f^{(n)}(0) + \ldots$$

Taylor's theorem for two independent variables

If we consider $z = f(x, y)$ where z is a function of two independent variables x and y, then, in general, increases in x and y will produce a combined increase in z.

So, if $z = f(x, y)$ then $z + \delta z = f(x+h, y+k)$



$h =$ increase in x
$k =$ increase in y.

For R: $f(x+h, y) = f(x, y) + hf_x(x, y) + \frac{h^2}{2!}f_{xx}(x, y) + \ldots$ (1)

where $f_x(x, y)$ denotes $\frac{\partial}{\partial x}f(x, y)$; $f_{xx}(x, y)$ denotes $\frac{\partial^2}{\partial x^2}f(x, y)$ etc.

From R to Q: $(x+h)$ is constant; y changes to $(y+k)$

$\therefore\ f(x+h, y+k) = f(x+h, y) + kf_y(x+h, y) + \frac{k^2}{2!}f_{yy}(x+h, y) + \ldots$ (2)

To express (2) in terms of $f(x, y)$ we can substitute result (1) for the first term $f(x+h, y)$ and similar expressions which we shall obtain for $f_y(x+h, y)$, $f_{yy}(x+h, y)$ and so on.

If we differentiate (1) with respect to y, we have

$$f_y(x+h, y) = \ldots\ldots\ldots\ldots$$

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$$f_y(x+h, y) = f_y(x, y) + hf_{yx}(x, y) + \frac{h^2}{2!}f_{yxx}(x, y) + \dots$$

If we now differentiate this result again with respect to y

$$f_{yy}(x+h, y) = \dots\dots\dots\dots$$

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$$f_{yy}(x+h,y) = f_{yy}(x,y) + hf_{yyx}(x,y) + \frac{h^2}{2!}f_{yyxx}(x,y) + \dots$$

Then our previous expansion (2), i.e.

$$f(x+h,y+k) = f(x+h,y) + kf_y(x+h,y) + \frac{k^2}{2!}f_{yy}(x+h,y) + \dots$$

now becomes

$$\begin{aligned} f(x+h, y+k) = f(x, y) &+ hf_x(x, y) + \frac{h^2}{2!}f_{xx}(x, y) + \dots \\ &+ k\left\{f_y(x, y) + hf_{yx}(x, y) + \frac{h^2}{2!}f_{yxx}(x, y) + \dots\right\} \\ &+ \frac{k^2}{2!}\left\{f_{yy}(x, y) + hf_{yyx}(x, y) + \frac{h^2}{2!}f_{yyxx}(x, y) + \dots\right\} \\ &+ \dots \end{aligned}$$

Rearranging the terms by collecting together all the first derivatives, and then all the second derivatives, and so on, we get

$$f(x+h, y+k) = \dots\dots\dots\dots$$

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$$\begin{aligned} f(x+h,y+k) = f(x,y) &+ \{hf_x(x,y) + kf_y(x,y)\} \\ &+ \frac{1}{2!}\{h^2f_{xx}(x,y) + 2hkf_{xy}(x,y) + k^2f_{yy}(x,y)\} + \dots \end{aligned}$$

This is Taylor's theorem for two independent variables.

Small increments

If $z = f(x, y)$, $h = \delta x$, $k = \delta y$, then Taylor's theorem can be written as

$$z + \delta z = z + \left\{h\frac{\partial z}{\partial x} + k\frac{\partial z}{\partial y}\right\} + \frac{1}{2!}\left\{h^2\frac{\partial^2 z}{\partial x^2} + 2hk\frac{\partial^2 z}{\partial y\,\partial x} + k^2\frac{\partial^2 z}{\partial y^2}\right\} + \dots$$

Subtracting z from each side:

$$\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y + \frac{1}{2!}\left\{\frac{\partial^2 z}{\partial x^2}(\delta x)^2 + 2\frac{\partial^2 z}{\partial y\,\partial x}(\delta x\,\delta y) + \frac{\partial^2 z}{\partial y^2}(\delta y)^2\right\} + \dots$$

Since δx and δy are small, the expression in the brackets is of the next order of smallness and can be discarded for our purposes. Therefore, we arrive at the result

$$\text{If}\quad z = f(x, y)\quad\text{then}\quad \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

As already explained above, this result is, in fact, an approximation since the smaller terms in the series have been neglected. For practical purposes, however, the result can be used as stated. **Be sure to make a note of the result, for it is the foundation of much that follows.**

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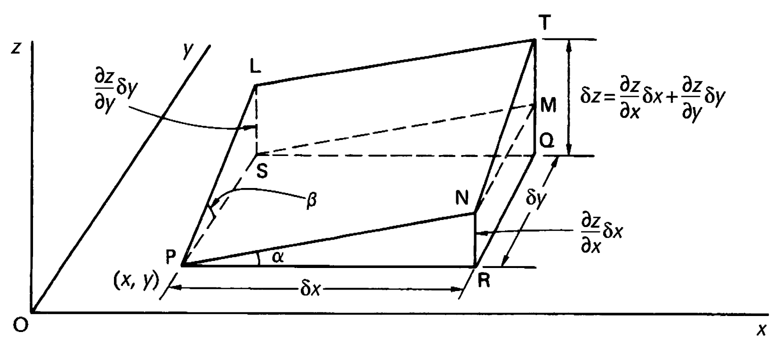
$$\boxed{z = f(x, y);\qquad \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y}$$

The following diagram illustrates the result.



$$\frac{\partial z}{\partial x}\text{ is the slope of PN}\quad \therefore\ \text{RN} = \frac{\partial z}{\partial x}\delta x = \text{QM}$$

$$\frac{\partial z}{\partial y}\text{ is the slope of PL}\quad \therefore\ \text{SL} = \frac{\partial z}{\partial y}\delta y = \text{MT}$$

$$\text{QT} = \text{QM} + \text{MT}\quad \therefore\ \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

This is the total increment of $z = f(x, y)$ from P to Q.

▶

It is worth noting at this stage that the result can be extended to the case of three independent variables, i.e. if $u = f(x, y, z)$

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$$

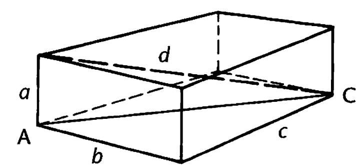
One or two straightforward applications will lay the foundations for future development.

Example

A rectangular box has sides measured as 30 mm, 40 mm and 60 mm. If these measurements are liable to be in error by $\pm 0{\cdot}5$ mm, $\pm 0{\cdot}8$ mm and $\pm 1{\cdot}0$ mm respectively, calculate the length of the diagonal of the box and the maximum possible error in the result.

First build up an expression for the diagonal d in terms of the sides, a, b and c.

$d = \ldots\ldots\ldots\ldots$



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$$\boxed{d = \sqrt{a^2 + b^2 + c^2}}$$

Because

$$d^2 = a^2 + AC^2 = a^2 + b^2 + c^2 \text{ and so } d = \sqrt{a^2 + b^2 + c^2}$$

Then $\delta d = \dfrac{\partial d}{\partial a}\delta a + \dfrac{\partial d}{\partial b}\delta b + \dfrac{\partial d}{\partial c}\delta c$

We now determine the partial differential coefficients and obtain an expression for δd, but all in terms of a, b and c. Do not yet insert numerical values.

$\delta d = \ldots\ldots\ldots\ldots$

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$$\boxed{\delta d = \frac{1}{\sqrt{a^2 + b^2 + c^2}}\{a\delta a + b\delta b + c\delta c\}}$$

Now, substituting the values $a = 30$, $b = 40$, $c = 60$

$\delta a = \pm 0{\cdot}5$, $\delta b = \pm 0{\cdot}8$, $\delta c = \pm 1{\cdot}0$

the calculated length of the diagonal $= \ldots\ldots\ldots\ldots$

the maximum possible error $= \ldots\ldots\ldots\ldots$

diagonal $= \sqrt{a^2 + b^2 + c^2} = 78{\cdot}10$ mm
maximum error $= \pm\, 1{\cdot}37$ mm

Because

$$\delta d = \frac{1}{78{\cdot}10}\{30(\pm\, 0{\cdot}5) + 40(\pm\, 0{\cdot}8) + 60(\pm\, 1{\cdot}0)\}$$

Greatest error when the signs are the same

$$\therefore\ \delta d = \frac{1}{78{\cdot}10}\{\pm\,(15 + 32 + 60)\} = \pm 1{\cdot}37 \text{ mm}$$

Rates of change

If $z = f(x, y)$, then we have seen that $\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$

Dividing through by δt: $\quad \frac{\delta z}{\delta t} = \frac{\partial z}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial z}{\partial y}\frac{\delta y}{\delta t}$

Then if $\delta t \to 0$: $\quad \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$

Note the result. Then on to an example.

Example

The base radius r of a right circular cone is increasing at the rate of $1{\cdot}5$ mm/s while the perpendicular height is decreasing at $6{\cdot}0$ mm/s. Determine the rate at which the volume V is changing when $r = 12$ mm and $h = 24$ mm.

Find an expression for $\frac{\mathrm{d}V}{\mathrm{d}t}$ in terms of r and h which is ....

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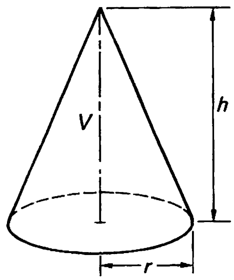
$$\boxed{\frac{dV}{dt}=\frac{2\pi rh}{3}\cdot\frac{dr}{dt}+\frac{\pi r^2}{3}\cdot\frac{dh}{dt}}$$



$$V=\frac{1}{3}\pi r^2 h;\quad \frac{dV}{dt}=\frac{\partial V}{\partial r}\cdot\frac{dr}{dt}+\frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$

$$\frac{\partial V}{\partial r}=\frac{2\pi rh}{3};\quad \frac{\partial V}{\partial h}=\frac{\pi r^2}{3}$$

$$\therefore\ \frac{dV}{dt}=\frac{2\pi rh}{3}\cdot\frac{dr}{dt}+\frac{\pi r^2}{3}\cdot\frac{dh}{dt}$$

Finally, we insert the numerical values:

$$r=12;\quad h=24;\quad \frac{dr}{dt}=1{\cdot}5;\quad \frac{dh}{dt}=-6{\cdot}0\quad (h\text{ is decreasing})$$

$$\frac{dV}{dt}=288\pi-288\pi=0$$

$\therefore$ At the instant when $r=12$ mm and $h=24$ mm,
the volume is unchanging.

Implicit functions

The same initial result, $\delta z=\dfrac{\partial z}{\partial x}\delta x+\dfrac{\partial z}{\partial y}\delta y$ enables us to determine the derivative of an implicit function $f(x,y)=0$, i.e. in a case where y is not defined explicitly in terms of x.

If $f(x,y)=0$ is an implicit function, we let $z=f(x,y)$.

Then, as before: $$\delta z=\frac{\partial z}{\partial x}\delta x+\frac{\partial z}{\partial y}\delta y$$

Dividing through by δx: $$\frac{\delta z}{\delta x}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{\delta y}{\delta x}$$

Then, if $\delta x\to 0$: $$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$$

But $z=0\ \therefore\ \dfrac{dz}{dx}=0$ $$\therefore\ \frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}=0$$

$$\therefore\ \frac{dy}{dx}=-\left(\frac{\partial z}{\partial x}\Big/\frac{\partial z}{\partial y}\right)$$

So, if $x^2-xy-y^2=0$, $\dfrac{dy}{dx}=$ ....

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$$\boxed{\frac{dy}{dx} = \frac{2x - y}{x + 2y}}$$

Putting $z = x^2 - xy - y^2$, $\dfrac{\partial z}{\partial x} = 2x - y$ and $\dfrac{\partial z}{\partial y} = -x - 2y$

The rest follows immediately.

Now on to the next frame

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The work so far, important though it is, is largely by way of revision of the more basic ideas of partial differentiation. We now extend these same ideas to further applications.

Change of variables

If $z = f(x, y)$ and x and y are themselves functions of two new independent variables, u and v, then we need expressions for $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Yet again, we start from the result we established at the beginning of this Programme.

If $z = f(x, y)$ then $\delta z = \dfrac{\partial z}{\partial x}\delta x + \dfrac{\partial z}{\partial y}\delta y$

Dividing in turn by δu and δv:

$$\frac{\delta z}{\delta u} = \frac{\partial z}{\partial x} \cdot \frac{\delta x}{\delta u} + \frac{\partial z}{\partial y} \cdot \frac{\delta y}{\delta u}$$

$$\frac{\delta z}{\delta v} = \frac{\partial z}{\partial x} \cdot \frac{\delta x}{\delta v} + \frac{\partial z}{\partial y} \cdot \frac{\delta y}{\delta v}$$

Then, as $\delta u \to 0$ and $\delta v \to 0$, these become

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

Example 1

If $z = x^2 - y^2$ and $x = r\cos\theta$ and $y = r\sin\theta$, then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}$$

and $$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

We now need the various partial derivatives

$$\frac{\partial z}{\partial x} = \ldots\ldots\ldots\ldots; \quad \frac{\partial x}{\partial r} = \ldots\ldots\ldots\ldots; \quad \frac{\partial y}{\partial r} = \ldots\ldots\ldots\ldots$$

$$\frac{\partial z}{\partial y} = \ldots\ldots\ldots\ldots; \quad \frac{\partial x}{\partial \theta} = \ldots\ldots\ldots\ldots; \quad \frac{\partial y}{\partial \theta} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\begin{array}{lll} \dfrac{\partial z}{\partial x} = 2x; & \dfrac{\partial x}{\partial r} = \cos\theta; & \dfrac{\partial y}{\partial r} = \sin\theta \\ \dfrac{\partial z}{\partial y} = -2y; & \dfrac{\partial x}{\partial \theta} = -r\sin\theta; & \dfrac{\partial y}{\partial \theta} = r\cos\theta \end{array}}$$

Substituting in the two equations and simplifying:

$$\frac{\partial z}{\partial r} = \ldots\ldots\ldots\ldots; \quad \frac{\partial z}{\partial \theta} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{\partial z}{\partial r} = 2x\cos\theta - 2y\sin\theta; \quad \frac{\partial z}{\partial \theta} = -(2xr\sin\theta + 2yr\cos\theta)}$$

Finally, we can express x and y in terms of r and θ as given, so that, after tidying up, we obtain

$$\frac{\partial z}{\partial r} = \ldots\ldots\ldots\ldots; \quad \frac{\partial z}{\partial \theta} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{\partial z}{\partial r} = 2r(\cos^2\theta - \sin^2\theta); \quad \frac{\partial z}{\partial \theta} = -4r^2\sin\theta\cos\theta}$$

Of course, we could express these as

$$\frac{\partial z}{\partial r} = 2r\cos 2\theta \quad \text{and} \quad \frac{\partial z}{\partial \theta} = -2r^2\sin 2\theta$$

From these results, we can, if necessary, find the second partial derivatives in the normal manner.

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right) = \frac{\partial}{\partial r}(2r\cos 2\theta) = 2\cos 2\theta$$

Similarly $$\frac{\partial^2 z}{\partial \theta^2} = \ldots\ldots\ldots\ldots \quad \text{and} \quad \frac{\partial^2 z}{\partial r\partial \theta} = \ldots\ldots\ldots\ldots$$

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$$\frac{\partial^2 z}{\partial \theta^2} = -4r^2 \cos 2\theta; \quad \frac{\partial^2 z}{\partial r \partial \theta} = -4r \sin 2\theta$$

Because

$$\frac{\partial^2 z}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial z}{\partial \theta}\right) = \frac{\partial}{\partial \theta}(-2r^2 \sin 2\theta) = -4r^2 \cos 2\theta$$

and $$\frac{\partial^2 z}{\partial r \partial \theta} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial \theta}\right) = \frac{\partial}{\partial r}(-2r^2 \sin 2\theta) = -4r \sin 2\theta$$

Example 2

If $z = f(x, y)$ and $x = \frac{1}{2}(u^2 - v^2)$ and $y = uv$, show that

$$u\frac{\partial z}{\partial v} - v\frac{\partial z}{\partial u} = 2\left(x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x}\right)$$

Although this is much the same as the previous example, there is, at least, one difference. In this case, we are not told the precise nature of $f(x, y)$. We must remember that z is a function of x and y and, therefore, of u and v. With that in mind, we set off with the usual two equations.

$$\frac{\partial z}{\partial u} = \dots\dots\dots\dots$$

$$\frac{\partial z}{\partial v} = \dots\dots\dots\dots$$

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$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

From the given information:

$$\frac{\partial x}{\partial u} = \dots\dots\dots\dots; \quad \frac{\partial y}{\partial u} = \dots\dots\dots\dots$$

$$\frac{\partial x}{\partial v} = \dots\dots\dots\dots; \quad \frac{\partial y}{\partial v} = \dots\dots\dots\dots$$

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$$\frac{\partial x}{\partial u} = u; \quad \frac{\partial y}{\partial u} = v$$
$$\frac{\partial x}{\partial v} = -v; \quad \frac{\partial y}{\partial v} = u$$

Whereupon $\dfrac{\partial z}{\partial u} = \ldots\ldots\ldots\ldots$

$\dfrac{\partial z}{\partial v} = \ldots\ldots\ldots\ldots$

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$$\frac{\partial z}{\partial u} = u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y}$$
$$\frac{\partial z}{\partial v} = -v\frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y}$$

If we now multiply the first of these by $(-v)$ and the second by u and add the two equations, we get the desired result.

$$-v\frac{\partial z}{\partial u} = -uv\frac{\partial z}{\partial x} - v^2\frac{\partial z}{\partial y}$$
$$u\frac{\partial z}{\partial v} = -uv\frac{\partial z}{\partial x} + u^2\frac{\partial z}{\partial y}$$

Adding $\quad u\dfrac{\partial z}{\partial v} - v\dfrac{\partial z}{\partial u} = -2uv\dfrac{\partial z}{\partial x} + (u^2 - v^2)\dfrac{\partial z}{\partial y}$

$$= -2y\frac{\partial z}{\partial x} + 2x\frac{\partial z}{\partial y}$$

$$\therefore\ u\frac{\partial z}{\partial v} - v\frac{\partial z}{\partial u} = 2\left(x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x}\right)$$

With the same given data, i.e.

$z = f(x, y)$ with $x = \dfrac{1}{2}(u^2 - v^2)$ and $y = uv$

we can now show that $\dfrac{\partial^2 z}{\partial u^2} + \dfrac{\partial^2 z}{\partial v^2} = (u^2 + v^2)\left(\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2}\right)$.

In determining the second partial derivatives, keep in mind that z is a function of u and v and that both of these variables also occur in $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

$$\frac{\partial^2 z}{\partial u^2} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{\partial^2 z}{\partial u^2} = \frac{\partial z}{\partial x} + u^2\frac{\partial^2 z}{\partial x^2} + 2uv\frac{\partial^2 z}{\partial x \partial y} + v^2\frac{\partial^2 z}{\partial y^2}}$$

Because

$$\frac{\partial}{\partial u} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial v} = \left(-v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}\right)$$

$$\therefore \quad \frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y}\right) = \frac{\partial z}{\partial x} + u\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) + v\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial z}{\partial x} + u\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\frac{\partial z}{\partial x} + v\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\frac{\partial z}{\partial y}$$

$$= \frac{\partial z}{\partial x} + u^2\frac{\partial^2 z}{\partial x^2} + uv\frac{\partial^2 z}{\partial x \partial y} + uv\frac{\partial^2 z}{\partial x \partial y} + v^2\frac{\partial^2 z}{\partial y^2}$$

$$\therefore \quad \frac{\partial^2 z}{\partial u^2} = \frac{\partial z}{\partial x} + u^2\frac{\partial^2 z}{\partial x^2} + 2uv\frac{\partial^2 z}{\partial x \partial y} + v^2\frac{\partial^2 z}{\partial y^2} \qquad (1)$$

Likewise, $\dfrac{\partial^2 z}{\partial v^2} = \dfrac{\partial}{\partial v}\left(\dfrac{\partial z}{\partial v}\right) = \dfrac{\partial}{\partial v}\left(-v\dfrac{\partial z}{\partial x} + u\dfrac{\partial z}{\partial y}\right)$

$= \dots\dots\dots\dots$

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$$\boxed{\frac{\partial^2 z}{\partial v^2} = \frac{\partial z}{\partial x} + v^2\frac{\partial^2 z}{\partial x^2} - 2uv\frac{\partial^2 z}{\partial x \partial y} + u^2\frac{\partial^2 z}{\partial y^2}}$$

Because

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(-v\frac{\partial z}{\partial x} + u\frac{\partial z}{\partial y}\right) = -\frac{\partial z}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial x}\right) + u\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial y}\right)$$

$$= \frac{\partial z}{\partial x} - v\left(-v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}\right)\frac{\partial z}{\partial x} + u\left(-v\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}\right)\frac{\partial z}{\partial y}$$

$$= \frac{\partial z}{\partial x} + v^2\frac{\partial^2 z}{\partial x^2} - uv\frac{\partial^2 z}{\partial x \partial y} - uv\frac{\partial^2 z}{\partial x \partial y} + u^2\frac{\partial^2 z}{\partial y^2}$$

$$\therefore \quad \frac{\partial^2 z}{\partial v^2} = \frac{\partial z}{\partial x} + v^2\frac{\partial^2 z}{\partial x^2} - 2uv\frac{\partial^2 z}{\partial x \partial y} + u^2\frac{\partial^2 z}{\partial y^2} \qquad (2)$$

Adding together results (1) and (2), we get

$\dots\dots\dots\dots$

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$$\boxed{\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = (u^2 + v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)}$$

and that is it.

Now, for something slightly different, move on to the next frame

Inverse functions

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If $z = f(x, y)$ and x and y are functions of two independent variables u and v defined by $u = g(x, y)$ and $v = h(x, y)$, we can theoretically solve these two equations to obtain x and y in terms of u and v. Hence we can determine $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$ and then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ as required.

In practice, however, the solution of $u = g(x, y)$ and $v = h(x, y)$ may well be difficult or even impossible by normal means. The following example shows how we can get over this difficulty.

Example 1

If $z = f(x, y)$ and $u = e^x \cos y$ and $v = e^{-x} \sin y$, we have to find $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$.

We start off once again with our standard relationships

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y \qquad (1)$$

$$\delta v = \frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y \qquad (2)$$

Now $u = e^x \cos y$ and $v = e^{-x} \sin y$

So $\quad \frac{\partial u}{\partial x} = \ldots\ldots\ldots\ldots; \quad \frac{\partial u}{\partial y} = \ldots\ldots\ldots\ldots$

$\quad \frac{\partial v}{\partial x} = \ldots\ldots\ldots\ldots; \quad \frac{\partial v}{\partial y} = \ldots\ldots\ldots\ldots$

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$$\boxed{\begin{array}{ll} \frac{\partial u}{\partial x} = e^x \cos y; & \frac{\partial u}{\partial y} = -e^x \sin y \\ \frac{\partial v}{\partial x} = -e^{-x} \sin y; & \frac{\partial v}{\partial y} = e^{-x} \cos y \end{array}}$$

Substituting in equations (1) and (2) above, we have

$$\delta u = e^x \cos y\, \delta x - e^x \sin y\, \delta y \qquad (3)$$

$$\delta v = -e^{-x} \sin y\, \delta x + e^{-x} \cos y\, \delta y \qquad (4)$$

Eliminating δy from (3) and (4), we get

$$\delta x = \ldots\ldots\ldots\ldots$$

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$$\boxed{\delta x = \frac{e^{-x}\cos y}{\cos 2y}\delta u + \frac{e^{x}\sin y}{\cos 2y}\delta v}$$

Because

$$(3) \times e^{-x}\cos y: \qquad e^{-x}\cos y\,\delta u = \cos^2 y\,\delta x - \sin y\cos y\,\delta y$$

$$(4) \times e^{x}\sin y: \qquad e^{x}\sin y\,\delta v = -\sin^2 y\,\delta x + \sin y\cos y\,\delta y$$

Adding: $e^{-x}\cos y\,\delta u + e^{x}\sin y\,\delta v = (\cos^2 y - \sin^2 y)\,\delta x$

$$\therefore\ \delta x = \frac{e^{-x}\cos y}{\cos 2y}\delta u + \frac{e^{x}\sin y}{\cos 2y}\delta v$$

But $\qquad \delta x = \dfrac{\partial x}{\partial u}\delta u + \dfrac{\partial x}{\partial v}\delta v$

$$\therefore\ \frac{\partial x}{\partial u} = \frac{e^{-x}\cos y}{\cos 2y} \quad \text{and} \quad \frac{\partial x}{\partial v} = \frac{e^{x}\sin y}{\cos 2y}$$

which are, of course, two of the expressions we have to find.

Starting again with equations (3) and (4), we can obtain

$$\delta y = \dots\dots\dots\dots$$

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$$\boxed{\delta y = \frac{e^{-x}\sin y}{\cos 2y}\delta u + \frac{e^{x}\cos y}{\cos 2y}\delta v}$$

Because

$$(3) \times e^{-x}\sin y: \qquad e^{-x}\sin y\,\delta u = \sin y\cos y\,\delta x - \sin^2 y\,\delta y$$

$$(4) \times e^{x}\cos y: \qquad e^{x}\cos y\,\delta v = -\sin y\cos y\,\delta x + \cos^2 y\,\delta y$$

Adding: $e^{-x}\sin y\,\delta u + e^{x}\cos y\,\delta v = (\cos^2 y - \sin^2 y)\,\delta y$

$$\therefore\ \delta y = \frac{e^{-x}\sin y}{\cos 2y}\delta u + \frac{e^{x}\cos y}{\cos 2y}\delta v$$

But, $\delta y = \dots\dots\dots\dots$ Finish it off.

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$$\delta y = \frac{\partial y}{\partial u}\delta u + \frac{\partial y}{\partial v}\delta v$$
$$\therefore \quad \frac{\partial y}{\partial u} = \frac{e^{-x}\sin y}{\cos 2y} \quad \text{and} \quad \frac{\partial y}{\partial v} = \frac{e^{x}\cos y}{\cos 2y}$$

So, collecting our four results together:

$$\frac{\partial x}{\partial u} = \frac{e^{-x}\cos y}{\cos 2y}; \quad \frac{\partial x}{\partial v} = \frac{e^{x}\sin y}{\cos 2y}$$
$$\frac{\partial y}{\partial u} = \frac{e^{-x}\sin y}{\cos 2y}; \quad \frac{\partial y}{\partial v} = \frac{e^{x}\cos y}{\cos 2y}$$

We can tackle most similar problems in the same way, but it is more efficient to investigate a general case and to streamline the results. Let us do that.

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General case

If $z = f(x, y)$ with $u = g(x, y)$ and $v = h(x, y)$, then we have

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y \qquad (1)$$

$$\delta v = \frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y \qquad (2)$$

We now solve these for δx and δy. Eliminating δy, we have

$(1) \times \frac{\partial v}{\partial y}$: $$\frac{\partial v}{\partial y}\delta u = \frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial x}\delta x + \frac{\partial v}{\partial y}\cdot\frac{\partial u}{\partial y}\delta y$$

$(2) \times \frac{\partial u}{\partial y}$: $$\frac{\partial u}{\partial y}\delta v = \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\delta x + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\delta y$$

Subtracting: $$\frac{\partial v}{\partial y}\delta u - \frac{\partial u}{\partial y}\delta v = \left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}\right)\delta x$$

$$\therefore \quad \delta x = \frac{\dfrac{\partial v}{\partial y}\delta u - \dfrac{\partial u}{\partial y}\delta v}{\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y} - \dfrac{\partial v}{\partial x}\cdot\dfrac{\partial u}{\partial y}}$$

Starting afresh from (1) and (2) and eliminating δx, we have

$$\delta y = \ldots\ldots\ldots\ldots$$

$$\boxed{\delta y = \frac{\dfrac{\partial u}{\partial x}\delta v - \dfrac{\partial v}{\partial x}\delta u}{\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y} - \dfrac{\partial v}{\partial x}\cdot\dfrac{\partial u}{\partial y}}}$$

The two results so far are therefore

$$\delta x = \frac{\dfrac{\partial v}{\partial y}\delta u - \dfrac{\partial u}{\partial y}\delta v}{\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y} - \dfrac{\partial v}{\partial x}\cdot\dfrac{\partial u}{\partial y}} \quad \text{and} \quad \delta y = \frac{\dfrac{\partial u}{\partial x}\delta v - \dfrac{\partial v}{\partial x}\delta u}{\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial v}{\partial y} - \dfrac{\partial v}{\partial x}\cdot\dfrac{\partial u}{\partial y}}$$

You will notice that the denominator is the same in each case and that it can be expressed in determinant form

$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

This determinant is called the *Jacobian* of u, v with respect to x, y and is denoted by the symbol J:

i.e. $$J = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$ and is often written as $\dfrac{\partial(u,v)}{\partial(x,y)}$

So $$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Our last two results can therefore be written

$$\delta x = \dots\dots\dots\dots; \quad \delta y = \dots\dots\dots\dots$$

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$$\delta x = \frac{\dfrac{\partial v}{\partial y}\delta u - \dfrac{\partial u}{\partial y}\delta v}{J} = \frac{\begin{vmatrix} \delta u & \delta v \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}}, \quad \delta y = \frac{\dfrac{\partial u}{\partial x}\delta v - \dfrac{\partial v}{\partial x}\delta u}{J} = \frac{\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \delta u & \delta v \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}}$$

We can now get a number of useful relationships.

(a) *If v is kept constant, $\delta v = 0$* $\quad \therefore \; \delta x = \dfrac{\partial v}{\partial y}\delta u \Big/ J$

Dividing by δu and letting $\delta u \to 0$ $\quad \dfrac{\partial x}{\partial u} = \dfrac{\partial v}{\partial y}\Big/ J$

Similarly $\quad \dfrac{\partial y}{\partial u} = -\dfrac{\partial v}{\partial x}\Big/ J$

(b) *If u is kept constant, $\delta u = 0$* $\quad \therefore \; \delta x = -\dfrac{\partial u}{\partial y}\delta v \Big/ J$

Dividing by δv and letting $\delta v \to 0$ $\quad \dfrac{\partial x}{\partial v} = -\dfrac{\partial u}{\partial y}\Big/ J$

Similarly $\quad \dfrac{\partial y}{\partial v} = \dfrac{\partial u}{\partial x}\Big/ J$

So, at this stage, we had better summarise the results.

Summary

If $z = f(x, y)$ and $u = g(x, y)$ and $v = h(x, y)$ then

$$\frac{\partial x}{\partial u} = \frac{\partial v}{\partial y}\Big/ J \qquad \frac{\partial x}{\partial v} = -\frac{\partial u}{\partial y}\Big/ J$$

$$\frac{\partial y}{\partial u} = -\frac{\partial v}{\partial x}\Big/ J \qquad \frac{\partial y}{\partial v} = \frac{\partial u}{\partial x}\Big/ J$$

where, in each case

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

Let us put this into practice by doing again the same example that we started with (Example 1 on page 382), but by the new method. First of all, however, make a note of the important summary listed above for future reference.

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Example 1A

If $z = f(x, y)$ and $u = e^x \cos y$ and $v = e^{-x} \sin y$, find the derivatives $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$.

$$u = e^x \cos y \qquad v = e^{-x} \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = -e^{-x} \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^{-x} \cos y$$

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \cos y & -e^{-x} \sin y \\ -e^x \sin y & e^{-x} \cos y \end{vmatrix}$$

$$= (e^x \cos y)(e^{-x} \cos y) - (-e^x \sin y)(-e^{-x} \sin y)$$

$$= \cos^2 y - \sin^2 y = \cos 2y$$

Then $$\frac{\partial x}{\partial u} = \frac{\partial v}{\partial y} \bigg/ J = \frac{e^{-x} \cos y}{\cos 2y}; \qquad \frac{\partial x}{\partial v} = -\frac{\partial u}{\partial y} \bigg/ J = \frac{e^x \sin y}{\cos 2y}$$

$$\frac{\partial y}{\partial u} = -\frac{\partial v}{\partial x} \bigg/ J = \frac{e^{-x} \sin y}{\cos 2y}; \qquad \frac{\partial y}{\partial v} = \frac{\partial u}{\partial x} \bigg/ J = \frac{e^x \cos y}{\cos 2y}$$

which is a lot shorter than our first approach.

Move on for a further example

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Example 2

If $z = f(x, y)$ with $u = x^2 - y^2$ and $v = xy$, find expressions for $\dfrac{\partial x}{\partial u}$, $\dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$.

First we need

$$\frac{\partial u}{\partial x} = \dots\dots\dots\dots; \quad \frac{\partial u}{\partial y} = \dots\dots\dots\dots; \quad \frac{\partial v}{\partial x} = \dots\dots\dots\dots; \quad \frac{\partial v}{\partial y} = \dots\dots\dots\dots$$

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$$\boxed{\frac{\partial u}{\partial x} = 2x; \quad \frac{\partial u}{\partial y} = -2y; \quad \frac{\partial v}{\partial x} = y; \quad \frac{\partial v}{\partial y} = x}$$

Then we calculate J which, in this case, is …………

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$$\boxed{J = 2(x^2 + y^2)}$$

Because

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & y \\ -2y & x \end{vmatrix} = 2x^2 + 2y^2$$

Finally, we have the four relationships

$$\frac{\partial x}{\partial u} = \frac{\partial v}{\partial y}\Big/ J = \dots\dots\dots\dots; \qquad \frac{\partial x}{\partial v} = -\frac{\partial u}{\partial y}\Big/ J = \dots\dots\dots\dots$$

$$\frac{\partial y}{\partial u} = -\frac{\partial v}{\partial x}\Big/ J = \dots\dots\dots\dots; \qquad \frac{\partial y}{\partial v} = \frac{\partial u}{\partial x}\Big/ J = \dots\dots\dots\dots$$

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$$\boxed{\begin{array}{ll} \dfrac{\partial x}{\partial u} = \dfrac{x}{2(x^2 + y^2)}; & \dfrac{\partial x}{\partial v} = \dfrac{y}{x^2 + y^2} \\ \dfrac{\partial y}{\partial u} = \dfrac{-y}{2(x^2 + y^2)}; & \dfrac{\partial y}{\partial v} = \dfrac{x}{x^2 + y^2} \end{array}}$$

And that is all there is to it.

If we know the details of the function $z = f(x, y)$ then we can go one stage further and use the results $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$ to find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Let us see this in a further example.

Example 3

If $z = 2x^2 + 3xy + 4y^2$ and $u = x^2 + y^2$ and $v = x + 2y$, determine

(a) $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$ (b) $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Section (a) is just like the previous example. Complete that on your own.

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$$\boxed{\frac{\partial x}{\partial u}=\frac{1}{2x-y};\quad \frac{\partial x}{\partial v}=\frac{-y}{2x-y};\quad \frac{\partial y}{\partial u}=\frac{-1}{2(2x-y)};\quad \frac{\partial y}{\partial v}=\frac{x}{2x-y}}$$

Because if $u=x^2+y^2$ and $v=x+2y$

$$\frac{\partial u}{\partial x}=2x;\ \frac{\partial u}{\partial y}=2y;\ \frac{\partial v}{\partial x}=1;\ \frac{\partial v}{\partial y}=2$$

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}2x & 1\\ 2y & 2\end{vmatrix}=4x-2y=2(2x-y)$$

Then $\dfrac{\partial x}{\partial u}=\dfrac{\partial v}{\partial y}\Big/J=2\Big/2(2x-y)=\dfrac{1}{2x-y}$

$$\frac{\partial x}{\partial v}=-\frac{\partial u}{\partial y}\Big/J=-2y\Big/2(2x-y)=\frac{-y}{2x-y}$$

$$\frac{\partial y}{\partial u}=-\frac{\partial v}{\partial x}\Big/J=-1\Big/2(2x-y)=\frac{-1}{2(2x-y)}$$

$$\frac{\partial y}{\partial v}=\frac{\partial u}{\partial x}\Big/J=2x\Big/2(2x-y)=\frac{x}{2x-y}$$

$$\therefore\ \frac{\partial x}{\partial u}=\frac{1}{2x-y};\quad \frac{\partial x}{\partial v}=\frac{-y}{2x-y};\quad \frac{\partial y}{\partial u}=\frac{-1}{2(2x-y)};\quad \frac{\partial y}{\partial v}=\frac{x}{2x-y}$$

Now for part (b).

Since z is also a function of u and v, the expressions for $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ are

$$\frac{\partial z}{\partial u}=\dots\dots\dots\dots$$
$$\frac{\partial z}{\partial v}=\dots\dots\dots\dots$$

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$$\boxed{\begin{aligned}\frac{\partial z}{\partial u}&=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\\ \frac{\partial z}{\partial v}&=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\end{aligned}}$$

The only remaining items of information we need are the expressions for $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ which we obtain from $z=2x^2+3xy+4y^2$

$$\frac{\partial z}{\partial x}=4x+3y \quad\text{and}\quad \frac{\partial z}{\partial y}=3x+8y$$

Using these and the previous set of derivatives, we now get

$$\frac{\partial z}{\partial u}=\dots\dots\dots\dots;\quad \frac{\partial z}{\partial v}=\dots\dots\dots\dots$$

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$$\boxed{\frac{\partial z}{\partial u} = \frac{5x - 2y}{2(2x - y)}; \quad \frac{\partial z}{\partial v} = \frac{3x^2 + 4xy - 3y^2}{2x - y}}$$

Because

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\therefore \quad \frac{\partial z}{\partial u} = (4x + 3y)\left\{\frac{1}{2x - y}\right\} + (3x + 8y)\left\{\frac{-1}{2(2x - y)}\right\}$$

$$= \frac{5x - 2y}{2(2x - y)} \qquad \therefore \quad \frac{\partial z}{\partial u} = \frac{5x - 2y}{2(2x - y)}$$

and $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x} \cdot \dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y} \cdot \dfrac{\partial y}{\partial v}$

$$\therefore \quad \frac{\partial z}{\partial v} = (4x + 3y)\left\{\frac{-y}{2x - y}\right\} + (3x + 8y)\left\{\frac{x}{2x - y}\right\}$$

$$= \frac{3x^2 + 4xy - 3y^2}{2x - y} \qquad \therefore \quad \frac{\partial z}{\partial v} = \frac{3x^2 + 4xy - 3y^2}{2x - y}$$

They are all done in the same general way.

Now on to the next topic

Stationary values of a function

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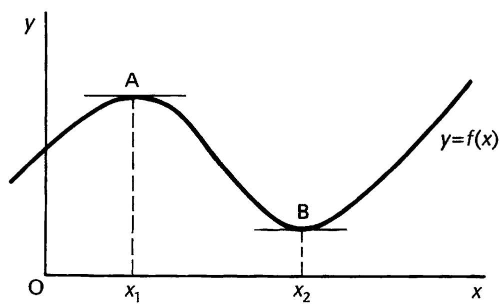
You will doubtless remember that in earlier work you established the characteristics of *stationary points* on a plane curve and derived the conditions that enable these critical points to be calculated.



At A and B $\qquad \dfrac{dy}{dx} = 0$

For maximum $\qquad \dfrac{d^2y}{dx^2}$ is negative $(x = x_1)$

For minimum $\qquad \dfrac{d^2y}{dx^2}$ is positive $(x = x_2)$

▶

We now progress to the application of these same considerations to three dimensions, where $z = f(x, y)$. The function is now represented by a surface and stationary values of the function $z = f(x, y)$ occur when the tangent plane to the surface at a point P (a, b) is parallel to the plane $z = 0$, i.e. to the x–y plane.

Let us take a closer look at this.

Maximum and minimum values

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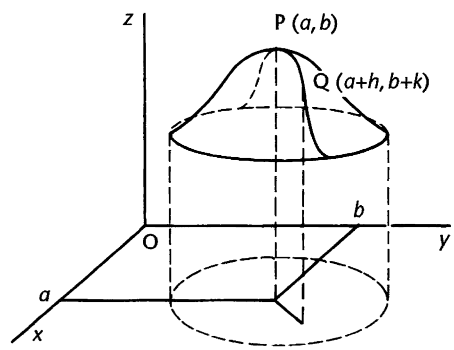
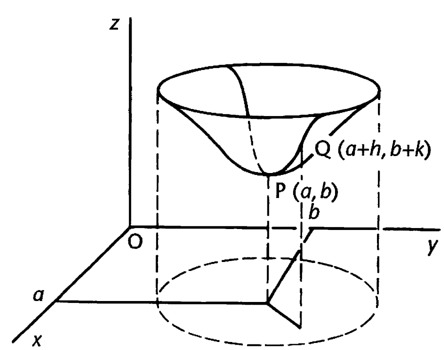


A function $z = f(x, y)$ is said to have *maximum* value at P (a, b) if $f(a, b)$ is greater than the value at a near-by point Q $(a + h, b + k)$ for all values of h and k however small, positive or negative, i.e. in all directions from P.



Similarly, $z = f(x, y)$ is said to have a *minimum* value at P (a, b) if $f(a, b)$ is less than the value at a neighbouring point Q $(a + h, b + k)$ in any direction from P.

To establish maximum and minimum values, we must therefore investigate the sign of the value of $f(a + h, b + k) - f(a, b)$.

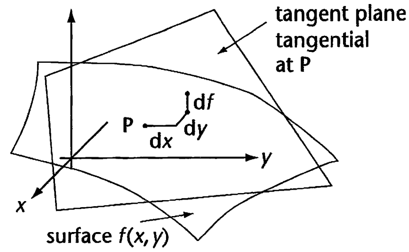
If $f(a + h, b + k) - f(a, b) < 0$ we have a maximum value at P (a, b).
If $f(a + h, b + k) - f(a, b) > 0$ we have a minimum value at P (a, b).

▶

To pursue this further we turn to the total differential

$$df(x, y) = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

The total differential measures the rise or fall in the tangent plane from the point of its contact with the surface at (x, y) to the point $(x + dx, y + dy)$.



If the point of contact is a maximum or a minimum then

$$\frac{\partial f}{\partial x} = \ldots\ldots\ldots\ldots \quad \text{and} \quad \frac{\partial f}{\partial y} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0}$$

Because

The tangent plane is parallel with the x–y plane and so the tangent plane neither rises nor falls, so that

$$df(x, y) = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0$$

Also because $dx \neq 0$ and $dy \neq 0$ then $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$.

Notice the logic here. If there is a maximum or a minimum, then $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. However, just because $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ at a point does not imply that a maximum or a minimum exists at that point. What we can say is that a *stationary* point exists at that point and, as we shall see later, not all stationary points are maxima or minima.

Example 1

Determine the values of x and y at which the stationary values of

$$f(x, y) = x^2 + xy + y^2 + 5x - 5y + 3$$

occur.

All we need to do is to obtain expressions for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, equate each to zero and then solve the pair of simultaneous equations so obtained. In which case

$$x = \ldots\ldots\ldots\ldots \quad \text{and} \quad y = \ldots\ldots\ldots\ldots$$

$$\boxed{x = -5 \quad \text{and} \quad y = 5}$$

Because

$\dfrac{\partial f}{\partial x} = 2x + y + 5$ and $\dfrac{\partial f}{\partial y} = x + 2y - 5$ giving the pair of simultaneous equations

$$2x + y + 5 = 0 \qquad (1)$$
$$x + 2y - 5 = 0 \qquad (2)$$

Adding (1) + (2) gives $3x + 3y = 0$, that is $y = -x$

Substitution in (1) gives $x = -5$ and so $y = 5$

Although a stationary value occurs at $(-5, 5)$ we have no evidence as to whether it is a maximum or a minimum value. Let us investigate further.

From the previous definitions

$f(a, b)$ will be a maximum value if $f(a + h, b + k) - f(a, b) < 0$

$f(a, b)$ will be a minimum value if $f(a + h, b + k) - f(a, b) > 0$

Now, from Taylor's theorem

$$f(a + h, b + k) = f(a, b) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots$$

and we have already seen that at a stationary value $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$. So, at a stationary point, Taylor's theorem becomes

$$f(a + h, b + k) - f(a, b) = \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots$$

where subsequent terms are of higher orders of h and k and are neglected.

The expression in the brackets on the right-hand side can be written as

$$\frac{1}{\dfrac{\partial^2 f}{\partial x^2}}\left\{\left(h\frac{\partial^2 f}{\partial x^2} + k\frac{\partial^2 f}{\partial x \partial y}\right)^2 + k^2\left(\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left[\frac{\partial^2 f}{\partial x \partial y}\right]^2\right)\right\}$$

Take a moment and expand the brackets and confirm that this is so.

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So $h^2\frac{\partial^2 f}{\partial x^2}+2hk\frac{\partial^2 f}{\partial x\partial y}+k^2\frac{\partial^2 f}{\partial y^2}$

$$=\frac{1}{\frac{\partial^2 f}{\partial x^2}}\left\{\left(h\frac{\partial^2 f}{\partial x^2}+k\frac{\partial^2 f}{\partial x\partial y}\right)^2+k^2\left(\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left[\frac{\partial^2 f}{\partial x\partial y}\right]^2\right)\right\}$$

Now $\left(h\frac{\partial^2 f}{\partial x^2}+k\frac{\partial^2 f}{\partial x\partial y}\right)^2$, being a square, is always positive and if $\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}>\left[\frac{\partial^2 f}{\partial x\partial y}\right]^2$ the second term will also be positive. In that case the sign of the whole expression is given by that of $\frac{\partial^2 f}{\partial x^2}$ at the front.

Furthermore, if $\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}>\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2$, i.e. $\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2>0$, this can be so only if $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ have the same sign. Therefore,

for $f(a,b)$ to be a maximum, $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are both negative

and for $f(a,b)$ to be a minimum, $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are both positive.

So, to determine whether a known stationary value is a maximum or a minimum value, we must find the second derivatives $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2 f}{\partial x\partial y}$.

Then

(a) If $\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2>0$, the stationary value is a true maximum or minimum value.

(b) In that case

(1) if $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are both *negative*, $f(a, b)$ is a *maximum*

(2) if $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are both *positive*, $f(a, b)$ is a *minimum*.

Make a careful note of the conclusions (a) and (b): then let us apply them.

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Example 2

Investigate further the stationary value of the function

$$z = x^2 + xy + y^2 + 5x - 5y + 3$$

We have already seen that this function has a stationary point at

$$x = \dots\dots\dots\dots; \quad y = \dots\dots\dots\dots$$

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$$\boxed{x = -5; \quad y = 5}$$

Next, we investigate the value of $\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$. If this is greater than zero at $(-5, 5)$, then either a maximum or a minimum occurs at that point.

Check whether this is so.

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$$\boxed{\text{Yes.} \quad \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0}$$

Because

$$\frac{\partial^2 z}{\partial x^2} = 2; \quad \frac{\partial^2 z}{\partial y^2} = 2; \quad \frac{\partial^2 z}{\partial x \partial y} = 1.$$

This confirms that $(-5, 5)$ is either a maximum or a minimum.

To decide which it is, we note that $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ are both *positive*.

$$\therefore \quad \text{at } (-5, 5), z \text{ is a } \dots\dots\dots\dots$$

46

$\boxed{\text{minimum}}$

Of course to find the actual minimum value of z we substitute $x = -5$ and $y = 5$ into the expression for z. That is really all there is to it. Another example.

Example 3

Determine the stationary values, if any, of the function

$z = x^3 - 6xy + y^3$

The four steps in the routine are:

(a) Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ and solve the equations $\dfrac{\partial z}{\partial x} = 0$ and $\dfrac{\partial z}{\partial y} = 0$.

(b) Determine whether $\left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) - \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$.

(c) If so, note the sign of $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$ to distinguish between max. and min.

(d) Evaluate the maximum or minimum value of z.

In this example, stationary values occur at ....

47

$$\boxed{z = 0 \text{ at } (0,\ 0) \quad \text{and} \quad z = -8 \text{ at } (2,\ 2)}$$

Because

$$z = x^3 - 6xy + y^3 \quad \therefore \quad \frac{\partial z}{\partial x} = 3x^2 - 6y \qquad \frac{\partial z}{\partial y} = -6x + 3y^2$$

$$\frac{\partial z}{\partial x} = 0 \text{ and } \frac{\partial z}{\partial y} = 0 \ \therefore \ x^2 - 2y = 0 \text{ and } -2x + y^2 = 0$$

A possible stationary point exists when $x^2 - 2y = 0$ and $-2x + y^2 = 0$. From the first equation $y = x^2/2$ and substitution into $-2x + y^2 = 0$ gives $-2x + x^4/4 = 0$.

That is $x^4 - 8x = x(x^3 - 8) = 0$ and so $x = 0$ or $x = 2$.

When $x = 0$ then $y = 0$ and when $x = 2$ then $y = 2$.

$\therefore$ There are stationary values at (0, 0) and (2, 2)

Next we determine whether $\left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) - \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$

Result ....

48

$$\boxed{\text{No max. or min. at } (0, 0); \quad \text{Either max. or min. at } (2, 2)}$$

Because

$$\frac{\partial z}{\partial x} = 3x^2 - 6y \qquad \therefore \quad \frac{\partial^2 z}{\partial x^2} = 6x$$

$$\frac{\partial z}{\partial y} = -6x + 3y^2 \qquad \therefore \quad \frac{\partial^2 z}{\partial y^2} = 6y$$

$$\frac{\partial^2 z}{\partial x \partial y} = -6$$

$$\therefore \quad \text{at } (0, 0) \quad \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (0)(0) - 36 < 0$$

$\therefore$ No max. or min. at (0, 0)

$$\text{At } (2, 2) \quad \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (12)(12) - 36 > 0$$

$\therefore$ Either max. or min. at (2, 2)

We see that at (2, 2) both $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ are positive. Therefore the stationary value at (2, 2) is a ....

49

$$\boxed{\text{minimum}}$$

Finally, the minimum value of z is ....

50

$$\boxed{-8}$$

Therefore, $\quad z_{\text{min}} = -8$ and occurs at (2, 2)

Before doing a further example, let us consider one other aspect of stationary values.

On to a new frame

51

Saddle point

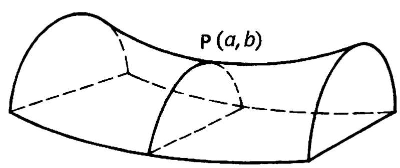
In the previous example, when we substituted the coordinates (0, 0) in the expression $\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$ we found that this did not satisfy the condition that for a maximum or minimum value

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$$

In fact, if $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$

and $$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 < 0$$

this is an indication of a form of stationary value described as a *saddle point*, as shown at P below.



A saddle point is, in effect, a combined maximum and minimum configuration in different directions. Its name is obvious from the shape.

Add this then to the list of conditions for stationary values that we have built up.

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At this stage, one naturally asks, what is implied if

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$$

In such a case, further detailed study of the function is necessary.

Now for an example to see it all in practice.

Example 4

Determine the stationary values of $z = 5xy - 4x^2 - y^2 - 2x - y + 5$.

Stationary values (or turning points) occur where

$$\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial z}{\partial y} = 0, \quad \text{i.e. at} \; \ldots\ldots\ldots\ldots$$

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$$\boxed{x = 1, \quad y = 2}$$

Because

$$\frac{\partial z}{\partial x} = 5y - 8x - 2 \qquad \frac{\partial z}{\partial y} = 5x - 2y - 1$$

$$\therefore \quad \left.\begin{aligned} 8x - 5y + 2 &= 0 \\ 5x - 2y - 1 &= 0 \end{aligned}\right\} \text{ gives } x = 1,\ y = 2$$

Therefore, the only stationary value occurs at (1, 2).
Next we substitute these x and y values in

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 \quad \text{and find} \ \ldots\ldots\ldots\ldots$$

54

$$\boxed{\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 < 0}$$

Because

$$\frac{\partial^2 z}{\partial x^2} = -8; \quad \frac{\partial^2 z}{\partial y^2} = -2; \quad \frac{\partial^2 z}{\partial x \partial y} = 5$$

$$\therefore \quad \left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = (-8)(-2) - 25 = -9 \quad \text{i.e.} < 0$$

The stationary value at (1, 2) is therefore a ....

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$$\boxed{\text{saddle point}}$$

Example 5

Determine stationary values of $z = x^3 - 3x + xy^2$ and their nature.

We go through the same routine as before.

First find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ and solve $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$.

Possible stationary values therefore occur at ....

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$$x = 0,\ y = \pm\sqrt{3};\quad x = \pm 1,\ y = 0$$

Because

$\dfrac{\partial z}{\partial x} = 3x^2 - 3 + y^2 \quad \dfrac{\partial z}{\partial y} = 2xy \quad \therefore\ x = 0$ or $y = 0$

If $x = 0$, $\quad y^2 = 3 \quad \therefore\ y = \pm\sqrt{3} \quad x = 0,\ y = \pm\sqrt{3}$

If $y = 0$, $\quad 3x^2 = 3 \quad \therefore\ x = \pm 1 \quad x = \pm 1,\ y = 0.$

Now we need the second derivatives and the usual tests. Finish if off. The nature of the stationary values:

$(0, \sqrt{3})$ ....; $\quad (0, -\sqrt{3})$ ....
$(1, 0)$ ....; $\quad (-1, 0)$ ....

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$(0, \sqrt{3})$ saddle point; $\quad (0, -\sqrt{3})$ saddle point
$(1, 0)$ minimum; $\quad (-1, 0)$ maximum

Because

$$\frac{\partial^2 z}{\partial x^2} = 6x; \quad \frac{\partial^2 z}{\partial y^2} = 2x; \quad \frac{\partial^2 z}{\partial x \partial y} = 2y$$

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$$

$(0, \sqrt{3})$	$(0)(0) - 12$	i.e. < 0	$\therefore$ saddle point
$(0, -\sqrt{3})$	$(0)(0) - 12$	i.e. < 0	$\therefore$ saddle point
$(1, 0)$	$(6)(2) - 0$	i.e. > 0	$\therefore$ minimum
$(-1, 0)$	$(-6)(-2) - 0$	i.e. > 0	$\therefore$ maximum

and that just about does everything.

Substitution of $(1, 0)$ and $(-1, 0)$ in $z = x^3 - 3x + xy^2$ gives the minimum and maximum values of z.

$z_{\min} = -2; \quad z_{\max} = 2.$

The value of z at each of the saddle points is zero.

Let's now look at some examples where the second derivative test fails

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Example 6

Determine the stationary values of $z = x^2 - 6xy + 9y^2$.

Here we see that $\dfrac{\partial z}{\partial x} = 2x - 6y$, $\dfrac{\partial z}{\partial y} = -6x + 18y$ and so these two derivatives vanish when

....

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$$\boxed{y = x/3}$$

Because

$\dfrac{\partial z}{\partial x} = 2x - 6y = 0$ when $2x = 6y$, that is when $y = x/3$ and

$\dfrac{\partial z}{\partial y} = -6x + 18y = 0$ when $6x = 18y$, that is when $y = x/3$, and so there is an infinity of stationary points lying along the line $y = x/3$.

Now

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = \ldots\ldots\ldots\ldots$$

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$$\boxed{0}$$

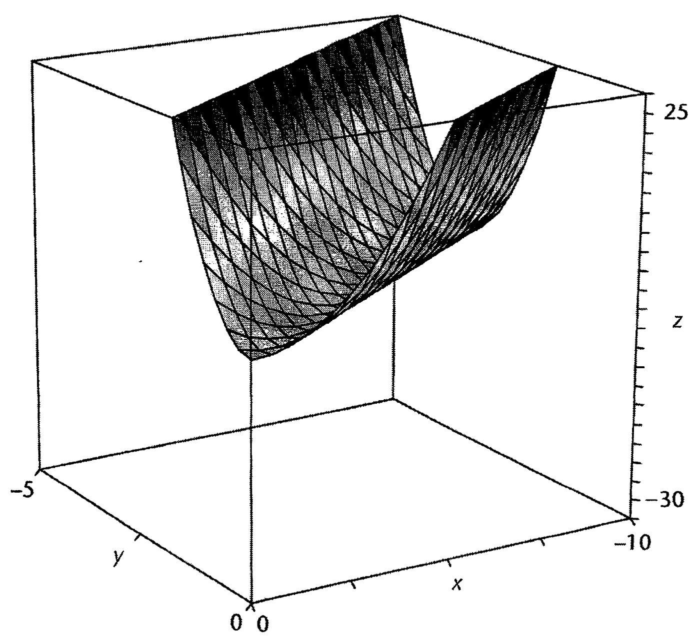
Because

$$\frac{\partial^2 z}{\partial x^2} = 2, \quad \frac{\partial^2 z}{\partial y^2} = 18 \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} = -6 \qquad \text{so that}$$

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 18 \times 2 - 36 = 0$$

So the second derivative test does not apply and we must look elsewhere to decide the nature of the stationary points.

Since $x^2 - 6xy + 9y^2 = (x - 3y)^2$ then $z \geq 0$ for all values of x and y. Therefore the stationary points are minima – there is an infinity of minimum points along the line $y = x/3$.



▶

Example 7

Find the stationary points of $z = x^4 - y^3$.

Here we see that $\dfrac{\partial z}{\partial x} = 4x^3$, $\dfrac{\partial z}{\partial y} = -3y^2$ and so these two derivatives vanish when $x = \dots\dots\dots\dots$, $y = \dots\dots\dots\dots$

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$$\boxed{x = 0, \quad y = 0}$$

Because

$\dfrac{\partial z}{\partial x} = 4x^3 = 0$ when $x = 0$ and $\dfrac{\partial z}{\partial y} = -2y^2 = 0$ when $y = 0$, so there is just one stationary point at (0, 0).

Now, at the stationary point

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = \dots\dots\dots\dots$$

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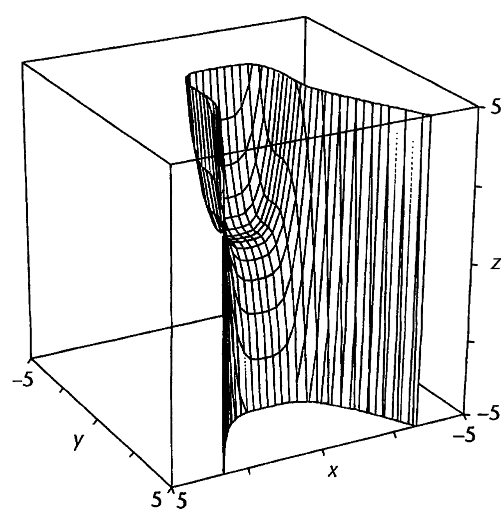
$$\boxed{0}$$

Because

$\dfrac{\partial^2 z}{\partial x^2} = 12x^2$, $\dfrac{\partial^2 z}{\partial y^2} = -4y$ and $\dfrac{\partial^2 z}{\partial x \partial y} = 0$ so that at (0, 0):

$$\left(\frac{\partial^2 z}{\partial x^2}\right)\left(\frac{\partial^2 z}{\partial y^2}\right) - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$$

So the second derivative test does not apply. However, in the z–x plane $y = 0$ and so $z = x^4$. This means that the line of intersection of the surface with the z–x plane has a minimum at the origin. In the z–y plane $x = 0$ and so $z = -y^3$. This means that the line of intersection of the surface with the z–y plane has a point of inflexion at the origin.



Lagrange undetermined multipliers

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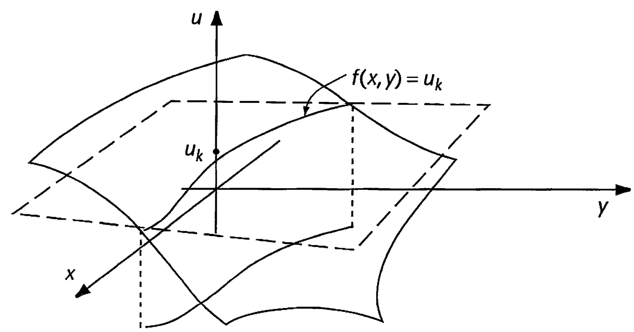
Closely allied to the problem of locating the stationary points of some function $u = f(x, y)$ is the problem of locating points where $u = f(x, y)$ attains its greatest or its least value (an extremal value) subject to the condition that x and y are related to each other via the equation

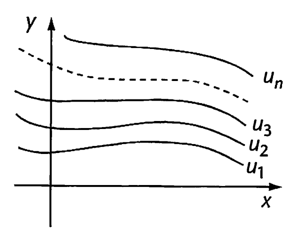
$$\phi(x, y) = 0 \qquad (1)$$

The problem can be clarified if we consider it graphically.

The graph of $u = f(x, y)$ is a surface within the (x, y, u) coordinate system. Selecting a plane parallel to the x–y plane on which the value of u is constant, u_k, we see that the surface intersects the plane in a curve given by the equation $f(x, y) = u_k$.



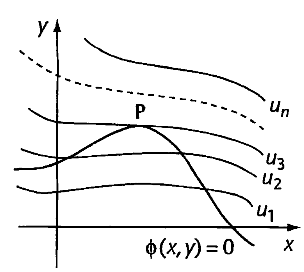
This line of intersection can now be projected onto the x–y plane to form what is known as a *level curve*. Different values of u_k determine different planes (all parallel to the x–y plane), different lines of intersection and hence different level curves. Accordingly, an alternative graphical description of $u = f(x, y)$ is that of a family of level curves in the x–y plane with each member of the family being associated with a particular value of u_k, where we assume that $u_1 < u_2 < u_3 < \ldots < u_n$ or $u_1 > u_2 > u_3 > \ldots > u_n$.



We now superimpose onto this family of level curves the graph of the constraint equation $\phi(x, y) = 0$.

Clearly, in the figure alongside, u_3 is the extremal value of $f(x, y)$ that coincides with $\phi(x, y) = 0$, and at the point P where they meet they share the same tangent line $\mathrm{d}y/\mathrm{d}x$. Now, since $\phi(x, y) = 0$ we see that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ldots\ldots\ldots\ldots$$



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$$\boxed{\frac{dy}{dx} = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y}}$$

Because

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = 0 \text{ so that } \frac{dy}{dx} = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y}$$

The same tangent can be found from

$$du = \frac{\partial f}{\partial x}dx = \frac{\partial f}{\partial y}dy$$

by equating the differential $du = 0$. Therefore

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y}$$

The latter two fractions are equivalent fractions which means that the two numerators and the two denominators each differ by the same multiplicative factor K, enabling us to say that

$$\frac{\partial f}{\partial x} = K\frac{\partial\phi}{\partial x} \text{ and } \frac{\partial f}{\partial y} = K\frac{\partial\phi}{\partial y} \text{ so that}$$

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial\phi}{\partial x} = 0 \qquad (2)$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial\phi}{\partial y} = 0 \qquad (3)$$

$\lambda = -K$ is called a Lagrange multiplier and equations (2) and (3), coupled with the constraint equation $\phi(x, y) = 0$, give us three relationships from which the values of x and y at the extremal points – and also the value of λ if required – can be found. Quite often the value of λ is not important.

Let us see how it works in a simple example.

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Example 1

Find the stationary points of the function $u = x^2 + y^2$ subject to the constraint $x^2 + y^2 + 2x - 2y + 1 = 0$.

In this case, $u = x^2 + y^2$

$\phi = x^2 + y^2 + 2x - 2y + 1$

We need to know

$$\frac{\partial u}{\partial x} = \dots\dots\dots\dots; \quad \frac{\partial u}{\partial y} = \dots\dots\dots\dots$$

$$\frac{\partial\phi}{\partial x} = \dots\dots\dots\dots; \quad \frac{\partial\phi}{\partial y} = \dots\dots\dots\dots$$

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$$\boxed{\frac{\partial u}{\partial x} = 2x; \quad \frac{\partial u}{\partial y} = 2y; \quad \frac{\partial \phi}{\partial x} = 2x + 2; \quad \frac{\partial \phi}{\partial y} = 2y - 2}$$

Then we form and solve
$$\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$

together with $\quad \phi = x^2 + y^2 + 2x - 2y + 1 = 0$

which gives $\quad x = \dots\dots\dots\dots; \; y = \dots\dots\dots\dots; \; \lambda = \dots\dots\dots\dots$

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$$\boxed{x = -1 \pm \frac{\sqrt{2}}{2}; \; y = 1 \mp \frac{\sqrt{2}}{2}; \; \lambda = \sqrt{2} - 1}$$

$$\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \quad \therefore \; 2x + \lambda(2x + 2) = 0 \quad \therefore \; x + \lambda(x + 1) = 0$$

$$\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \quad \therefore \; 2y + \lambda(2x - 2) = 0 \quad \therefore \; y + \lambda(y - 1) = 0$$

$$\therefore \; \frac{x}{y} = \frac{-\lambda(x+1)}{-\lambda(y-1)} \quad \therefore \; xy - x = xy + y \quad \therefore \; y = -x$$

Substituting this in ϕ

$$x^2 + x^2 + 2x + 2x + 1 = 0 \qquad 2x^2 + 4x + 1 = 0$$

$$\therefore \; x = -1 \pm \frac{\sqrt{2}}{2}$$

But $y = -x$ $\quad \therefore \; y = 1 \mp \dfrac{\sqrt{2}}{2}$

To find λ, we have $x + \lambda(x + 1) = 0 \quad \therefore \; \lambda = \sqrt{2} \mp 1$

As we have already said, we do not really need to find the value of λ.

On to the next

Functions with three independent variables

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The argument is very much the same as before.

To find stationary points of the function $\quad u = f(x, y, z)$ $\qquad$ (1)

subject to the constraint $\quad \phi(x, y, z) = 0$ $\qquad$ (2)

Again we have, at stationary points

$$\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z = 0 \qquad (3)$$

and since $\phi(x, y, z) = 0$

then
$$\frac{\partial \phi}{\partial x}\delta x + \frac{\partial \phi}{\partial y}\delta y + \frac{\partial \phi}{\partial z}\delta z = 0 \qquad (4)$$

Multiplying each term in (4) by λ and adding (4) to (3), we have

$\dots\dots\dots\dots$

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$$\boxed{\left(\frac{\partial u}{\partial x}+\lambda\frac{\partial \phi}{\partial x}\right)\delta x+\left(\frac{\partial u}{\partial y}+\lambda\frac{\partial \phi}{\partial y}\right)\delta y+\left(\frac{\partial u}{\partial z}+\lambda\frac{\partial \phi}{\partial z}\right)\delta z=0}$$

from which $\frac{\partial u}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0$ (5)

$\frac{\partial u}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0$ (6)

$\frac{\partial u}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0$ (7)

Equations (5), (6), (7), together with the constraint (2), provide all the information to determine x, y, z, and, if necessary, λ.

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Example 2

To find the stationary points of the function

$u=x^2+2y^2+z$

subject to the constraint $\phi(x,z)=x^2-z^2-2=0$.

So $\frac{\partial u}{\partial x}=\dots\dots\dots\dots;\quad \frac{\partial u}{\partial y}=\dots\dots\dots\dots;\quad \frac{\partial u}{\partial z}=\dots\dots\dots\dots$

$\frac{\partial \phi}{\partial x}=\dots\dots\dots\dots;\quad \frac{\partial \phi}{\partial y}=\dots\dots\dots\dots;\quad \frac{\partial \phi}{\partial z}=\dots\dots\dots\dots$

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$\frac{\partial u}{\partial x}=2x;$	$\frac{\partial u}{\partial y}=4y;$	$\frac{\partial u}{\partial z}=1$
$\frac{\partial \phi}{\partial x}=2x;$	$\frac{\partial \phi}{\partial y}=0;$	$\frac{\partial \phi}{\partial z}=-2z$

Now compile the equations

$$\frac{\partial u}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0;\quad \frac{\partial u}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0;\quad \frac{\partial u}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0$$

and, together with the constraint $\phi=x^2-z^2-2=0$, establish that stationary points occur at ....

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$$\boxed{\left(\tfrac{3}{2},\ 0,\ -\tfrac{1}{2}\right)\ \text{and}\ \left(-\tfrac{3}{2},\ 0,\ -\tfrac{1}{2}\right)}$$

Because

$$\frac{\partial u}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0\quad \therefore\ 2x+\lambda 2x=0\quad \therefore\ \lambda=-1$$

$$\frac{\partial u}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0\quad 4y+\lambda(0)=0\quad \therefore\ y=0$$

$$\frac{\partial u}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0\quad 1-\lambda 2z=0\quad \therefore\ z=\frac{1}{2\lambda}=-\frac{1}{2}$$

$$\phi=x^2-z^2-2=0\quad \therefore\ x^2-\tfrac{1}{4}-2=0\quad \therefore\ x=\pm\tfrac{3}{2}.$$

Therefore, stationary points at $\left(\frac{3}{2},\ 0,\ -\frac{1}{2}\right)$ and $\left(-\frac{3}{2},\ 0,\ -\frac{1}{2}\right)$.

▶

The method of Lagrange multipliers does not lend itself easily to give a distinction between the various types of stationary points. In many practical applications, however, whether a result is a maximum or a minimum value will be apparent from the physical consideration of the problem.

Let us finish with one further example.

So move on

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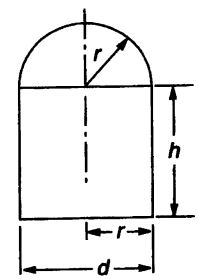
Example 3

A hot water storage tank is a vertical cylinder surmounted by a hemispherical top of the same diameter. The tank is designed to hold 400 m^3 of liquid. Determine the total height and the diameter of the tank if the surface heat loss is to be a minimum.



We first write down the function for the total surface area, A.

$$A = \ldots\ldots\ldots\ldots$$

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$$\boxed{A = 3\pi r^2 + 2\pi rh}$$

Because

The surface area of the hemisphere is $2\pi r^2$, the area of the base of the tank is πr^2 and the area of the cylindrical side is $2\pi rh$, giving a total area of $3\pi r^2 + 2\pi rh$.

This is the function which has to be a minimum. The constraint in this problem is that …………

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$$\boxed{\text{the volume is } 400 \text{ m}^3}$$

So we have

$$A = 3\pi r^2 + 2\pi rh \quad (1)$$

constraint $\quad V = \pi r^2 h + \frac{2}{3}\pi r^3 = 400$

So let $\quad \phi = \pi r^2 h + \frac{2}{3}\pi r^3 - 400 = 0 \quad (2)$

We now want $\quad \dfrac{\partial A}{\partial r} = \ldots\ldots\ldots\ldots; \quad \dfrac{\partial A}{\partial h} = \ldots\ldots\ldots\ldots$

$$\frac{\partial \phi}{\partial r} = \ldots\ldots\ldots\ldots; \quad \frac{\partial \phi}{\partial h} = \ldots\ldots\ldots\ldots$$

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$$\frac{\partial A}{\partial r} = 6\pi r + 2\pi h; \quad \frac{\partial \phi}{\partial r} = 2\pi rh + 2\pi r^2$$
$$\frac{\partial A}{\partial h} = 2\pi r; \quad \frac{\partial \phi}{\partial h} = \pi r^2$$

Now we form $\quad \dfrac{\partial A}{\partial r} + \lambda\dfrac{\partial \phi}{\partial r} = 0$

and $\quad \dfrac{\partial A}{\partial h} + \lambda\dfrac{\partial \phi}{\partial h} = 0$

and, with the constraint, $\phi = \pi r^2 h + \frac{2}{3}\pi r^3 - 400 = 0$,
we eventually obtain $r =$ .... and $h =$ ....
Finish it off and hence find the total height and the diameter.

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$$r = 4{\cdot}243 \text{ m}; \quad h = 4{\cdot}243 \text{ m}$$

Check the working:

$$\frac{\partial A}{\partial r} + \lambda\frac{\partial \phi}{\partial r} = 0 \quad \therefore \quad 6\pi r + 2\pi h + \lambda(2\pi rh + 2\pi r^2) = 0 \qquad (3)$$

$$\frac{\partial A}{\partial h} + \lambda\frac{\partial \phi}{\partial h} = 0 \quad \therefore \quad 2\pi r + \lambda\pi r^2 = 0 \qquad (4)$$

From (4): $\quad \lambda = -\dfrac{2}{r} \quad$ Substitute this in (3)

$$6\pi r + 2\pi h - \frac{2}{r}(2\pi rh + 2\pi r^2) = 0$$
$$\therefore \; 6r + 2h - 4h - 4r = 0 \qquad \therefore \; h = r$$

Also $\quad \pi r^2 h + \dfrac{2}{3}\pi r^3 = 400 \quad \therefore \; \dfrac{5}{3}\pi r^3 = 400 \quad \therefore \; r = 4{\cdot}243$

$\therefore$ Total height $= h + r = 8{\cdot}49$ m; Diameter $= 8{\cdot}49$ m

That brings us to the end of this particular Programme and to the usual **Revision summary** that follows. Check through the **Can You?** checklist and be sure to revise any section should you feel that is necessary. Then you will find the **Test exercise** straightforward – no tricks. The **Further problems** provide valuable additional practice.

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Revision summary 10

1 *Small increments*

$$z = f(x,y) \quad \delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$$

$$u = f(x,y,z) \quad \delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$$

2 *Rates of change*

$$z = f(x,y) \quad \frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

3 *Implicit functions*

$$z = f(x,y) = 0 \quad \frac{dy}{dx} = -\left(\frac{\partial z}{\partial x}\Big/\frac{\partial z}{\partial y}\right)$$

4 *Change of variables*

$z = f(x,y)$ x and y are functions of u and v

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

5 *Inverse functions*

$z = f(x,y) \quad u = g(x,y) \quad v = h(x,y)$

$$\frac{\partial x}{\partial u} = \frac{\partial v}{\partial y}\Big/J; \quad \frac{\partial x}{\partial v} = -\frac{\partial u}{\partial y}\Big/J$$

$$\frac{\partial y}{\partial u} = -\frac{\partial v}{\partial x}\Big/J; \quad \frac{\partial y}{\partial v} = \frac{\partial u}{\partial x}\Big/J$$

where $J = \dfrac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial v}{\partial x} \\ \dfrac{\partial u}{\partial y} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

6 *Stationary points*

$z = f(x,y)$ (a) $\dfrac{\partial z}{\partial x} = 0$ and $\dfrac{\partial z}{\partial y} = 0$

(b) $\left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) - \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$ for max. or min.

$\left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) - \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 < 0$ for saddle point

$\left(\dfrac{\partial^2 z}{\partial x^2}\right)\left(\dfrac{\partial^2 z}{\partial y^2}\right) - \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$ no decision without further information

(c) $\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$ both *negative* for maximum

$\dfrac{\partial^2 z}{\partial x^2}$ and $\dfrac{\partial^2 z}{\partial y^2}$ both *positive* for minimum.

▶

7 *Lagrange multipliers*
Two independent variables
$u = f(x, y)$ with constraint $\phi(x, y) = 0$

Solve
$$\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$
with
$$\phi(x, y) = 0.$$

Three independent variables
$u = f(x, y, z)$ with constraint $\phi(x, y, z) = 0$

Solve
$$\frac{\partial u}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0$$
$$\frac{\partial u}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0$$
with
$$\phi(x, y, z) = 0.$$

Can You?

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Checklist 10

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Derive the expression for a small increment in an expression of two real variables using Taylor's theorem? 1 to 4
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Apply the notion of small increments in expressions in two and three real variables to a variety of problems? 5 to 7
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Determine the rate of change with respect to time of an expression involving two or three real variables? 8 and 9
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Differentiate implicit functions? 9 and 10
  Yes ☐ ☐ ☐ ☐ ☐ *No*
- Determine first and second derivatives involving change of variables in expressions of two real variables? 11 to 21
  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Use the Jacobian to obtain the derivatives of inverse functions of two real variables? **22** to **37**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Locate and identify maxima, minima and saddle points of functions of two real variables? **38** to **57**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Solve problems where the independent variables are constrained by using the method of Lagrange undetermined multipliers for functions of two and three real variables? **58** to **77**

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 10

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1 If $z = \dfrac{xy}{x-y}$, show that

(a) $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$

(b) $x^2\dfrac{\partial^2 z}{\partial x^2} - y^2\dfrac{\partial^2 z}{\partial y^2} = 0$

(c) $z\dfrac{\partial^2 z}{\partial x \partial y} = 2\dfrac{\partial z}{\partial x}.\dfrac{\partial z}{\partial y}$.

2 Two sides of a triangular plate are measured as 125 mm and 160 mm, each to the nearest millimetre. The included angle is quoted as $60° \pm 1°$. Calculate the length of the remaining side and the maximum possible error in the result.

3 If $z = \left(x^2 - y^2\right)^{1/2}$ and x is increasing at 3·5 m/s, determine at what rate y must change in order that z shall be neither increasing nor decreasing at the instant when $x = 5$ m and $y = 3$ m.

4 If $2x^2 + 4xy + 3y^2 = 1$, obtain expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

5 If $u = x^2 + y^2$ and $v = 4xy$, determine

$\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$.

6 Determine the position and nature of the stationary points of the functions:

(a) $z = 2x^2y^2 + 4xy^2 - 4y^3 + 16y + 5$

(b) $z = 4 - 25x^2 + 20xy - 4y^2$.

▶

7 A rectangular storage tank is to have a capacity of $1{\cdot}0\ \text{m}^3$. If the tank is closed and the top is made of metal half as thick as the sides and base, use Lagrange's method of undetermined multipliers to determine the dimensions of the tank for the total amount of metal used in its construction to be a minimum.

8 Use Lagrange's method of undetermined multipliers to obtain the stationary values of $u = x^2 + y^2 + z^2$ subject to the constraint $\phi = 3x - 2y + z - 4$.

Further problems 10

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1 If $z = 2x^2 - 3y$ with $u = x^2 \sin y$ and $v = 2y \cos x$, determine expressions for $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

2 If $u = x^2 + e^{-3y}$ and $v = 2x + e^{3y}$, determine $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$.

3 If $z = f(x, y)$ where $x = uv$ and $y = u^2 - v^2$, show that

(a) $2x\dfrac{\partial z}{\partial x} + 2y\dfrac{\partial z}{\partial y} = u\dfrac{\partial z}{\partial u} + v\dfrac{\partial z}{\partial v}$

(b) $2\dfrac{\partial z}{\partial y} = \dfrac{1}{u^2 + v^2}\left\{u\dfrac{\partial z}{\partial u} - v\dfrac{\partial z}{\partial v}\right\}$.

4 If $V = f(x, y)$ and $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

5 If $z = \cosh 2x \sin 3y$ and $u = e^x(1 + y^2)$ and $v = 2ye^{-x}$, determine expressions for $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}$, and hence find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

6 If $z = f(u, v)$ where $u = \frac{1}{2}(x^2 - y^2)$ and $v = xy$, prove that

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 2u\left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}\right) + 4v\frac{\partial^2 z}{\partial u \partial v} + 2\frac{\partial z}{\partial u}.$$

7 Locate the stationary points of the following functions. Determine the nature of the points and calculate the critical function values.

(a) $z = y^2 + xy + x^2 + 4y - 4x + 5$

(b) $z = y^2 + xy + 2x + 3y + 6$

(c) $z = 3xy - 6y^2 - 3x^2 + 6y + 6x + 7$.

8 Find the stationary points of the function

$$z = (x^2 + y^2)^2 - 8(x^2 - y^2)$$

and determine their nature.

9 Verify that the function $z = (x + y - 1)/(x^2 + 2y^2 + 2)$ has stationary values at (2, 1) and $\left(-\frac{2}{3}, -\frac{1}{3}\right)$ and determine their nature.

10 Locate stationary points of the function

$$z = 4x^2 + 10xy + 4y^2 - x^2y^2$$

and determine their nature.

11 Find the stationary points of the following functions and determine their nature.

(a) $z = x(x^2 - 3) + 3y(x - 1)^2 + 18y^2(2y - 3)$

(b) $z = x^2y^2 - x^2 - y^2$.

12 Find the stationary points of the following functions and determine their nature.

(a) $z = (x - y)(x^2 + xy + y^2)$

(b) $z = 6 - x^2 + 8xy - 16y^2$

(c) $z = \cos(x^2 + y^2)$.

13 A metal channel is formed by turning up the sides of width x of a rectangular sheet of metal through an angle θ. If the sheet is 200 mm wide, determine the values of x and θ for which the cross-section of the channel will be a maximum.

14 A container is in the form of a right circular cylinder of length l and diameter d, with equal conical ends of the same diameter and height h. If V is the fixed volume of the container, find the dimensions l, h and d for minimum surface area.

15 A solid consists of a cylinder of length l and diameter d, surmounted at one end by a cone of vertex angle 2θ and base diameter d, and at the other end by a hemisphere of the same diameter. If the volume V of the solid is 50 cm^3, determine the dimensions l, d and θ so that the total surface area shall be a minimum.

16 A rectangular solid of maximum volume is to be cut from a solid sphere of radius r. Determine the dimensions of the solid so formed and its volume.

17 Use Lagrange's method of undetermined multipliers to obtain the stationary values of the following functions u, subject in each case to the constraint ϕ.

(a) $u = x^2y^2z^2 \qquad \phi = x^2 + y^2 + z^2 - 4 = 0$

(b) $u = x^2 + y^2 \qquad \phi = 4x^2 + 6xy + 4y^2 = 9$.